1.a $\quad x(t) = Ae^{(\sigma + j\omega)t}$

$$Re(x) = \frac{1}{2}(x + x^*) = \frac{1}{2}\left(Ae^{(\sigma+j\omega)t} + \left(Ae^{(\sigma+j\omega)t}\right)^*\right)$$
$$= \frac{1}{2}\left(Ae^{(\sigma+j\omega)t} + Ae^{(\sigma-j\omega)t}\right)$$
$$= \frac{1}{2}\left(Ae^{\sigma t}e^{j\omega t} + Ae^{\sigma t}e^{-j\omega t}\right)$$
$$= \frac{1}{2}Ae^{\sigma t}\left(e^{j\omega t} + e^{-j\omega t}\right)$$
$$= \boxed{Ae^{\sigma t}\cos(\omega t)}$$

$$Im(x) = \frac{1}{2j}(x - x^*) = \frac{1}{2j}\left(Ae^{\sigma t}e^{j\omega t} - Ae^{\sigma t}e^{-j\omega t}\right)$$
$$= Ae^{\sigma t}\frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$
$$= \boxed{Ae^{\sigma t}\sin(\omega t)}$$

1.b

$$|x|^2 = xx^* = Ae^{(\sigma+j\omega)t}Ae^{(\sigma-j\omega)t} = A^2\left(e^{\sigma t}\right)^2 e^{j(\omega-\omega)t}$$
$$= A^2 e^{2\sigma t}$$
$$\boxed{|x| = Ae^{\sigma t}}$$

$\boxed{\arg(x) = \omega t}$, because $x = |x|e^{j\arg(x)} = Ae^{\sigma t}e^{j\underbrace{\omega t}_{\arg(x)}}$

1.c

$$herm(x) = \frac{1}{2}(x + x^\dagger) = \frac{1}{2}\left(Ae^{\sigma t}e^{j\omega t} + \left(Ae^{-\sigma t}e^{-j\omega t}\right)^*\right)$$
$$= \frac{A}{2}\left(e^{\sigma t}e^{j\omega t} + e^{-\sigma t}e^{+j\omega t}\right) = \frac{A}{2}\left(e^{\sigma t} + e^{-\sigma t}\right)e^{j\omega t}$$
$$= A\cosh(\sigma t)e^{j\omega t} = \boxed{A\cosh(\sigma t)\cos(\omega t) + jA\cosh(\sigma t)\sin(\omega t)}$$

$$antiherm(x) = \frac{1}{2}(x - x^\dagger) = \frac{1}{2}\left(Ae^{\sigma t}e^{j\omega t} - Ae^{-\sigma t}e^{+j\omega t}\right)$$
$$= \frac{A}{2}\left(e^{\sigma t} - e^{-\sigma t}\right)e^{j\omega t} = A\sinh(\sigma t)e^{j\omega t} = \boxed{A\sinh(\sigma t)\cos(\omega t) + jA\sinh(\sigma t)\sin(\omega t)}$$

2.a

$y = x^*$

Non linear: $(jx)^* = -jx \neq jy$ (but addition still works: $y_1+y_2 = x_1^*+x_2^* = (x_1+x_2)^*$)

Time invariant: $x(t) \xrightarrow{\text{delay}} x(t-\tau) \xrightarrow{\text{system}} x^*(t-\tau)$

$x(t) \xrightarrow{\text{system}} x^*(t) \xrightarrow{\text{delay}} x^*(t-\tau)$

equal

2.b

Linear: $\int_{-1}^{1} \underset{\text{odd}}{x_o(t)}\, dt = \underset{\text{symmetric}}{\int_{-1}^{1}} \frac{1}{2}(x(t) - x(-t))\, dt = \frac{1}{2}\int_{-1}^{1} x(t)\, dt - \frac{1}{2}\int_{-1}^{1} x(-t)\, dt$

$\lambda = -t$

$d\lambda = -dt$

$= \frac{1}{2}\int_{-1}^{1} x(t)dt + \frac{1}{2}\int_{1}^{-1} x(\lambda)\, d\lambda$

$= \frac{1}{2}\int_{-1}^{1} x(t)dt + -\frac{1}{2}\int_{-1}^{1} x(\lambda)\, d\lambda = 0$

So, $y(t) = x(t) + \int_{-1}^{1} x_o(t)dt = x(t) + 0 = x(t)$

obviously, $y_1+y_2 = x_1+x_2$ and $\alpha y = \alpha x$, so it is linear.

Time Invariant: $x(t) \xrightarrow{\text{delay}} x(t-\tau) \xrightarrow{\text{system}} x(t-\tau) + \int_{-1}^{1} \underbrace{\text{odd}(x(t-\tau))}_{\text{still odd, still zero}}\, dt = x(t-\tau)$

$x(t) \xrightarrow{\text{system}} x(t) + \int_{-1}^{1} x_o(t)dt = x(t) \xrightarrow{\text{delay}} x(t-\tau)$

2.c

Linear: from the definition of multiplication in GF(2), $0\cdot 0 = 0$ and $1 \cdot 1 = 1$. These are the only possible values, thus $x^2 = x\cdot x = x$.

$y = x$ is linear.

Time invariant: $x \xrightarrow{\text{delay}} x(t-\tau) \xrightarrow{\text{system}} x^2(t-\tau)$

$x(t) \xrightarrow{\text{system}} x^2(t) \xrightarrow{\text{delay}} x^2(t-\tau)$

equal

2.d Note, this system is just finding (2 times) the odd part of the input

Linear:

$$y_1 = x_1(t) - x_1(-t) \qquad y_2(t) = x_2(t) - x_2(-t)$$

$$x(t) = x_1(t) + x_2(t)$$

$$y(t) = x(t) - x(-t) = (x_1(t) + x_2(t)) - (x_1(-t) + x_2(-t))$$

$$= (x_1(t) - x_1(-t)) + (x_2(t) - x_2(-t))$$

$$= y_1(t) + y_2(t) = y(t)$$

$$\alpha x(t) - \alpha x(-t) = \alpha(x(t) - x(-t)) = \alpha(y(t))$$

time varying:

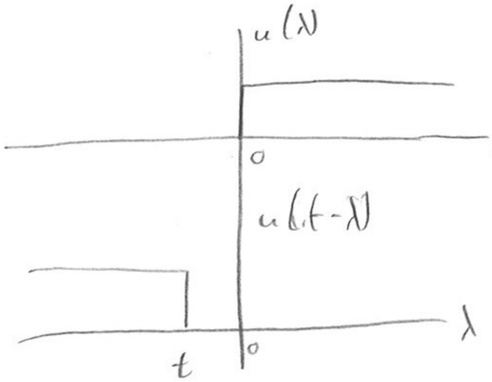
$$x(t) \xrightarrow{\text{delay}} x(t-\tau) \xrightarrow{\text{system}} x(t-\tau) - x(-t-\tau)$$

$$x(t) \xrightarrow{\text{system}} x(t) - x(-t) \xrightarrow{\text{delay}} x(t-\tau) - x(-(t-\tau))$$

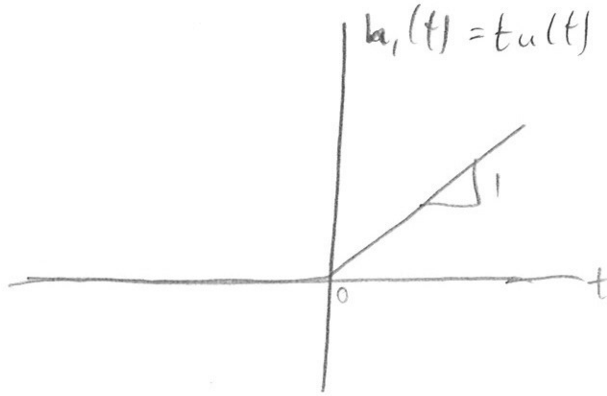
$$= x(t-\tau) - x(-t+\tau)$$

3.a

$$u * u = \int_{-\infty}^{\infty} u(\lambda)\, u(t-\lambda)\, d\lambda = \int_0^{\infty} u(t-\lambda)\, d\lambda = \begin{cases} 0 & t<0 \\ t & t>0 \end{cases}$$

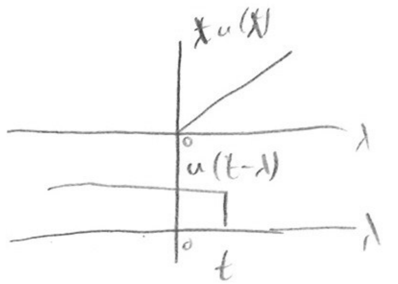


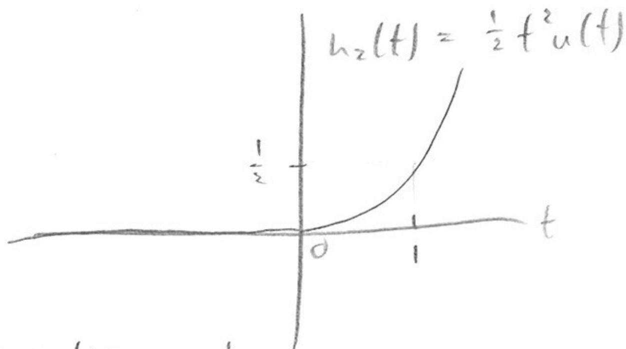
$$= \boxed{t\,u(t)}$$



3.b

$$h_1 * u = \int_{-\infty}^{\infty} \lambda\, u(\lambda)\, u(t-\lambda)\, d\lambda = \int_0^{\infty} \lambda\, u(t-\lambda)\, d\lambda = \begin{cases} 0 & t<0 \\ \frac{t^2}{2} & t>0 \end{cases} = \boxed{\frac{1}{2} t^2 u(t)}$$





3.c Note: convolution by u is an antiderivative operator,

the pattern should be clear:

$$h_0 = 1\, u(t)$$
$$h_1 = t\, u(t)$$
$$h_2 = \tfrac{1}{2} t^2 u(t)$$

Guess: $\boxed{\frac{1}{n!} t^n u(t) = h_n(t)}$

proof by induction:

$$h_0 = \frac{1}{0!} t^0 u(t) = u(t) \quad \text{OK}$$

$$h_{n+1} = h_n * u = \int_{-\infty}^{\infty} \frac{1}{n!} \lambda^n u(\lambda)\, u(t-\lambda)\, d\lambda = \int_0^{\infty} \frac{1}{n!} \lambda^n u(t-\lambda)\, d\lambda = \begin{cases} 0 & t<0 \\ \frac{1}{n!}\frac{1}{n+1} t^{n+1} & t>0 \end{cases}$$

$$= \frac{1}{n!}\frac{1}{n+1} t^{n+1} u(t) = \frac{1}{(n+1)!} t^{n+1} u(t) = h_{n+1}(t)$$

4.a)

$$y(t) = \int_{t-1}^{t} (1+\lambda)\,x(\lambda)\,d\lambda - t\int_{t-1}^{t} x(\lambda)\,d\lambda$$

$$= \int_{t-1}^{t} \big((1+\lambda)\,x(\lambda) - t\,x(\lambda)\big)\,d\lambda$$

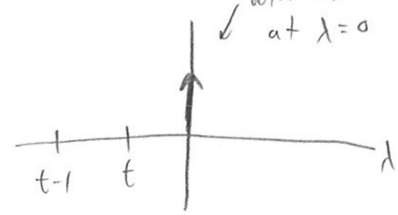
$$= \int_{t-1}^{t} \big(1-(t-\lambda)\big)\,x(\lambda)\,d\lambda \qquad \begin{aligned} \sigma &= t-\lambda \\ d\sigma &= -d\lambda \\ \lambda &= t-\sigma \end{aligned}$$

$$= -\int_{t-(t-1)}^{t-t} (1-\sigma)\,x(t-\sigma)\,d\sigma$$

$$= -\int_{1}^{0} (1-\sigma)\,x(t-\sigma)\,d\sigma = \int_{0}^{1} (1-\sigma)\,x(t-\sigma)\,d\sigma$$

$$= \int_{\cancel{\infty}}^{\infty} \underbrace{\Pi(\sigma-1)(\sigma-1)}_{h(\sigma)}\,x(t-\sigma)\,d\sigma = (h * x)(t)$$

OR, just set $x = \delta$

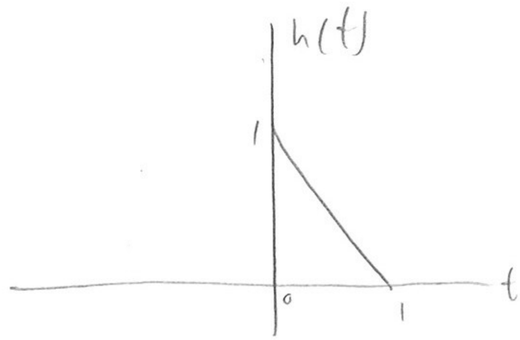


$$h(t) = \int_{t-1}^{t} (1+\lambda)\,\delta(\lambda) - t\int_{t-1}^{t} \delta(\lambda)\,d\lambda$$

$$= \begin{cases} 0 & t<0 \\ (1+0) + -t\cdot 1 = 1-t & t>0 \text{ and } t-1<0 \\ 0 & t-1>0 \end{cases}$$

$$= \begin{cases} 0 & t<0 \\ 1-t & 0<t<1 \\ 0 & t>1 \end{cases}$$

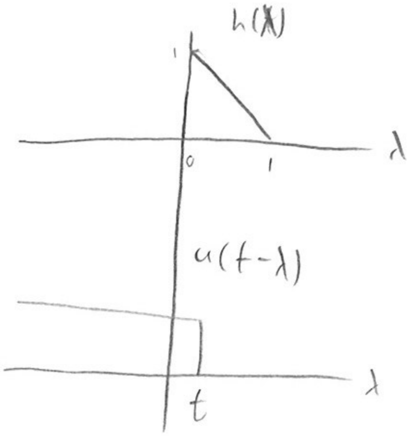
$$= \boxed{(1-t)\,\Pi(t-1)}$$



4.b

step response $= h * u$

$$= \int_{-\infty}^{\infty} h(\lambda)\, u(t-\lambda)\, d\lambda$$



$t<0$: $y = 0$

$0<t<1$: $\int_0^t (1-\lambda)\,d\lambda = -\frac{(1-\lambda)^2}{2}\Big]_{\lambda=0}^{t} = -\frac{(1-t)^2}{2} + \frac{1}{2} = \frac{1}{2}\left(1-(1-2t+t^2)\right)$

$$= t - \frac{t^2}{2}$$

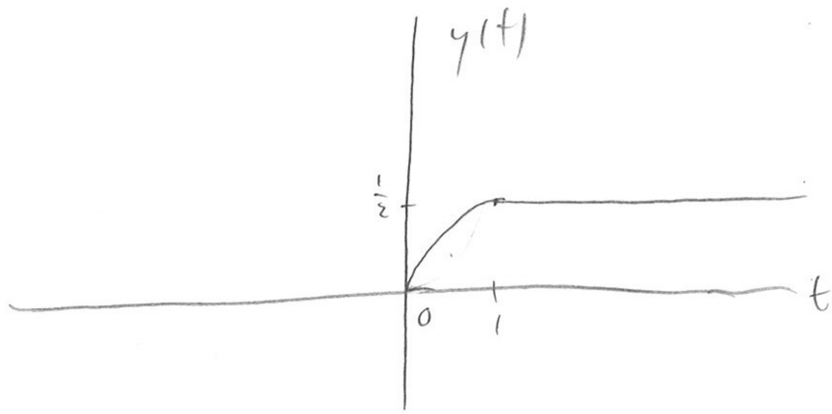
$t>1$: $\int_0^1 (1-\lambda)\,d\lambda = \underbrace{\tfrac{1}{2}\cdot 1\cdot 1}_{\text{area of triangle}} = \frac{1}{2}$

$$\boxed{y(t) = \begin{cases} 0 & t<0 \\ t-\frac{t^2}{2} & 0<t<1 \\ \frac{1}{2} & t>1 \end{cases}}$$

check: $y(t)$ is continuous:

$y(0) = 0 - \frac{0^2}{2} = 0$

$y(1) = 1 - \frac{1^2}{2} = 1 - \frac{1}{2} = \frac{1}{2}$



5]

T_1 and T_3 are orthogonal iff $\langle T_1, T_3 \rangle = 0$

$$\langle T_1, T_3 \rangle = \int_{-1}^{1} t(4t^3 - 3t) \frac{dt}{\sqrt{1-t^2}}$$

$$= \int_{-1}^{1} \frac{4t^4 - 3t^2}{\sqrt{1-t^2}} dt$$

$$= \cancel{\tfrac{3}{2}\sin^{-1}(t)} - (t^3 + \cancel{\tfrac{3}{2}t})\sqrt{1-t^2} - \cancel{\tfrac{3}{2}\sin^{-1}(t)} + \cancel{\tfrac{3}{2}t\sqrt{1-t^2}} \Big]_{t=-1}^{1}$$

$$= -t^3\sqrt{1-t^2}\Big]_{t=-1}^{1} = 0 - 0 = 0 \rightarrow \underline{\text{orthogonal}}$$

Note: These are Chebyshev polynomials. They are important for filter design and many other things.